7. Naći dužinu krive $y = \ln x$ od tačke sa apsicisom 1 do tačke sa apscisom $\sqrt{3}$.

Rešenje:

$$\ell = \int\limits_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x}\, dx = \int\limits_{\pi/4}^{\pi/3} \frac{dz}{\sin z \cos^2 z} = \left(\frac{1}{\cos z} + \ln\left|\operatorname{tg}\frac{z}{2}\right|\right)\Bigg|_{\pi/4}^{\pi/2} =$$

$$= 2 - \sqrt{2} - \frac{1}{2}\ln 3 - \ln \operatorname{tg}\frac{\pi}{8}.$$

8. Izračunati dužinu astroide $x = a\cos^3 t,\ y = a\sin^3 t, \quad t \in [0, 2\pi]$, $a > 0$.

Rešenje: Koristeći simetriju dobija se da je

$$\ell = 4\int_0^{\pi/2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 12a \int_0^{\pi/2} \sin t \cos t \; dt = 6a.$$

9. Izračunati dužinu luka prvog zavoja Arhimedove[12] spirale $r = a\varphi$, $0 \le \varphi \le 2\pi, \quad a > 0.$

Rešenje: $\ell = \displaystyle\int_0^{2\pi} \sqrt{a^2\varphi^2 + a^2}\, d\varphi = a\int_0^{2\pi} \sqrt{1+\varphi^2}\, d\varphi =$

$$= a\left(\pi\sqrt{1+4\pi^2} + \frac{1}{2}\ln\left(2\pi + \sqrt{1+4\pi^2}\right)\right).$$

$$l = \int_a \sqrt{1 + (f'(x))^2}\, dx$$

Dužina luka

(87) Naći dužinu luka koji na paraboli $y^2 = 2x + 1$ odsjeca prava $x - y = 1 \rightarrow y = x - 1$

Jednostavnije $x(y)$

$\rightarrow$ presjek sa ~~Oy osom~~ Ox osom

$y = 0 \rightarrow 2x + 1 = 0 \rightarrow 2x = -1 \Rightarrow x = -\frac{1}{2}$ $\quad A\left(-\frac{1}{2}, 0\right)$

$2x = -y^2 + 1$

Tjeme $\rightarrow \left(-\frac{D}{4a}, -\frac{b}{2a}\right)$ $\quad x = -\frac{1}{2}y^2 + \frac{1}{2}$

$b = 0 \Rightarrow$ presjek sa Ox osom tjeme $\rightarrow \boxed{T\left(-\frac{1}{2}, 0\right)}$

$\rightarrow x = 0 \Leftrightarrow y^2 = 1 \rightarrow y = \pm 1$ $\quad \boxed{A_1(0,1)}$ $\boxed{A_2(0,-1)}$

presjek parabole i prave

$y^2 = 2x + 1$

$x = +\frac{1}{2}y^2 + \frac{1}{2}$

$x = y + 1$

$+\frac{1}{2}y^2 + \frac{1}{2} = y + 1$

$\frac{1}{2}y^2 + y + \frac{1}{2} = 0 \,/\cdot 2$

$y_{1,2} =$

lakše:

$y^2 = 2x + 1$

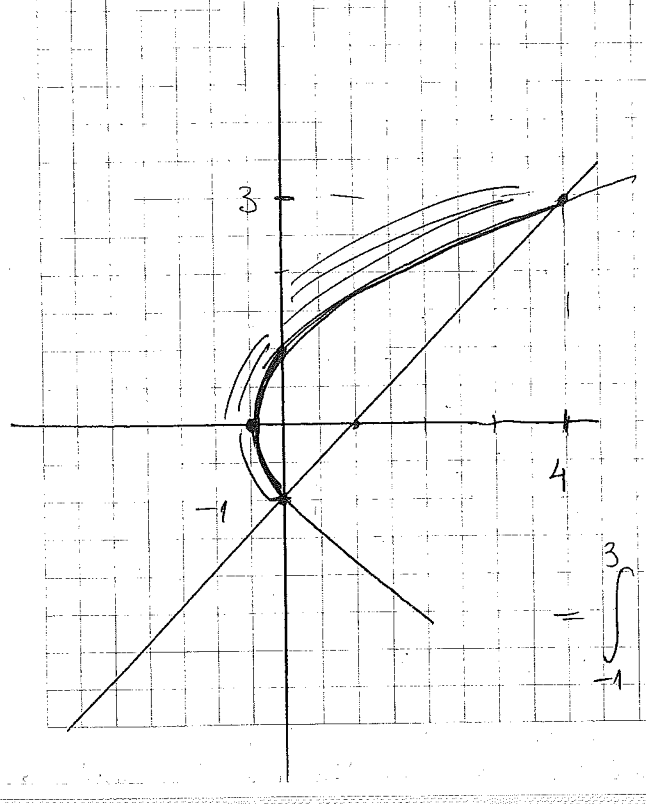
$y = x - 1$

$(x-1)^2 = 2x + 1$

$\hookrightarrow x_{1,2} = 0, 4$

$x = 0 \Rightarrow y = -1$ $\quad \boxed{B_1(0,-1)}$

$x = 4 \Rightarrow y = 3$ $\quad \boxed{B_2(4,3)}$



Jednostavnije

~~$y = f$~~ $x = f(y)$

~~$l = \int_{-\frac{1}{2}}$~~

$2x = y^2 - 1$

$x = \frac{1}{2}y^2 - \frac{1}{2}$ $\quad f'(y) = y$

$$l = \int_{-1}^{3} \sqrt{1 + (f'(y))^2}\, dy =$$

$$= \int_{-1}^{3} \sqrt{1 + y^2}\, dy$$

$$l=\int_{-1}^{3}\sqrt{1+y^2}\,dy=\left[\begin{array}{l} U=\sqrt{1+y^2}\Rightarrow dU=\frac{1}{\cancel{2}}\frac{\cancel{2}y}{\sqrt{1+y^2}}dy \\ dV=dy\Rightarrow V=\int dy=y\end{array}\right]=$$

$$=\cancel{\int_{-1}^{3}}=y\cdot\sqrt{1+y^2}\Big|_{-1}^{3}-\int_{-1}^{3}\frac{y^2+1-1}{\sqrt{1+y^2}}dy=$$

$$=y\cdot\sqrt{1+y^2}\Big|_{-1}^{3}-\underbrace{\int_{-1}^{3}\sqrt{1+y^2}\,dy}_{l}+\int_{-1}^{3}\frac{dy}{\sqrt{1+y^2}}$$

$$l=3\sqrt{10}+\sqrt{2}-l+\ln\left|y+\sqrt{y^2+1}\right|\Big|_{-1}^{3}$$

$$2l=\ldots\;;\quad l=(\ldots)\cdot\frac{1}{2}$$

88. Naći dužinu luka krive $\begin{cases} x=\frac{t^6}{6} \\ y=2-\frac{t^4}{4}\end{cases}$

između presječnih tačaka sa koordinatnim osama

dužina luka parametarski $l=\int_{\alpha}^{\beta}\sqrt{(x'(t))^2+(y'(t))^2}\,dt$

Presjek sa Oy osom $\rightarrow x=0$ Presjek sa Ox osom $\rightarrow y=0$

$\frac{t^6}{6}=0 \rightarrow t=0 \rightarrow A(0,2)$ $\quad 2-\frac{t^4}{4}=0$

$y=2-\frac{0}{4}=2$ $\quad t^4=8\rightarrow t=\sqrt[4]{8}$

$x'(t)=t^5$

$y'(t)=-t^3$ $\quad l=\int_{t_1}^{t_2}\sqrt{x'^2(t)+y'^2(t)}\,dt=\int_{0}^{\sqrt[4]{8}}\sqrt{t^{10}+t^6}\,dt=$

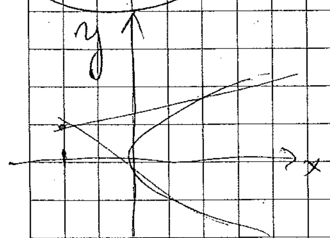
$$=\int_0^{\sqrt[4]{8}}t^3\sqrt{t^4+1}\,dt=\left[\begin{array}{l} t^4+1=z \\ 4t^3\,dt=dz \\ t^3dt=\frac{1}{4}dz\end{array}\quad\begin{array}{c|c|c} t & 0 & \sqrt[4]{8} \\ z & 1 & 9\end{array}\right]=$$

$$=\frac{1}{4}\int_1^9\sqrt{z}\,dz=\frac{1}{4}\cdot\frac{2}{3}\sqrt{z^3}\Big|_1^9=\frac{26}{6}=\frac{13}{3}$$

89) Iz tačke $P(-2,1)$ van parabole $y^2=4x$ povučene su tangente na nju

a) Naći j-ne tih tangenti

b) Izračunati dužinu luka od tjemena parabole do one tačke dodira sa tangentom koja ima veću apscisu $\longrightarrow$ x koordinata



Neka je $y=kx+n$ j-na tangente iz tačke $P(-2,1)$ na parabolu $y^2=4x$,

Tačka P pripada tangenti pa je:

$$1 = k\cdot(-2)+n \longrightarrow -2k+n=1$$

Naćimo presjek tangente i parabole

$$\begin{cases} y^2=4x \\ y=kx+n \end{cases} \qquad (kx+n)^2=4x \qquad k^2x^2+2kxn+n^2=4x \qquad k^2x^2+2(kn-2)x+n^2=0$$

Kako tangenta dodiruje krivu u jednoj tački, to posledinja j-na ima samo jedno rješ. tj $D=0$

$a=k^2$ $\qquad D=b^2-4ac$

$b=2(kn-2)$ $\qquad 4(kn-2)^2-4k^2n^2=0$

$c=n^2$ $\qquad \cancel{k^2n^2}-4kn+4-\cancel{k^2n^2}=0$

$$kn=1$$

$$\begin{cases} -2k+n=1 \\ kn=1 \end{cases} \Big/ \cdot(k) \longrightarrow \begin{cases} -2k^2+kn=k \\ 2k^2+k-1=0 \end{cases}$$

$$k_{1,2}=\frac{-1\pm\sqrt{1+8}}{4}; \qquad k_{1,2}=\frac{-1\pm3}{4} \qquad k_1=\frac{1}{2} \quad n_1=2 \qquad k_2=-1 \quad n_2=-1$$

$$t_1:\ y=\frac{1}{2}x+2 \qquad t_2:\ y=-x-1$$

Tačka dodira parabole i t_1

$$\begin{cases} y^2=4x \\ y=\frac{1}{2}x+2 \end{cases} \longrightarrow A(4,4)$$

prvi izvod f-je u tački

1. izvod -1

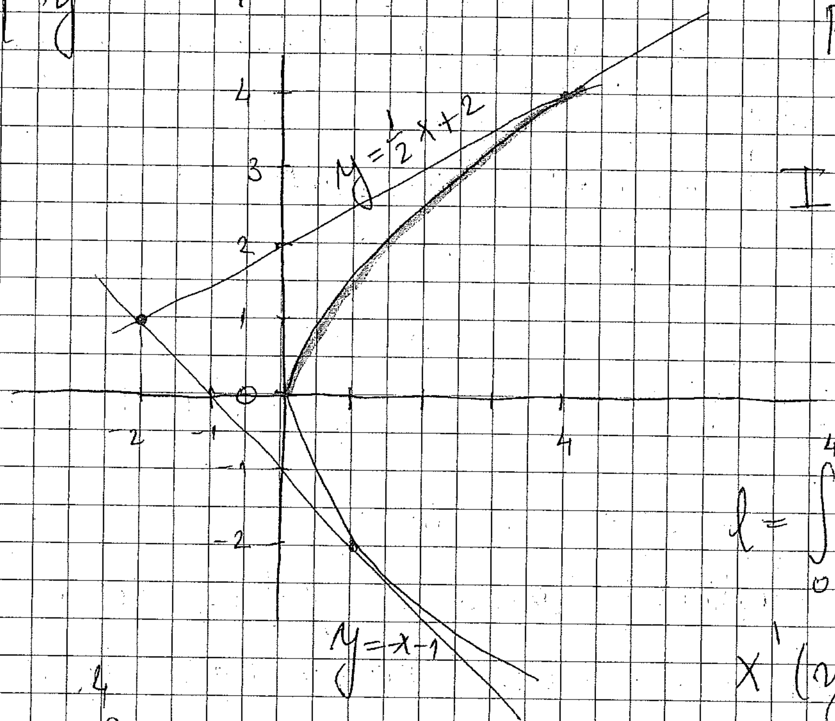
1. izvod $\frac{1}{2}$

Tačka dodira parabole i t_2

$$\begin{cases} y^2=4x \\ y=-x-1 \end{cases} \longrightarrow B(1,-2)$$

$A(4,4) \to$ Apscisa 4

$B(1,-2) \to$ Apscisa 1



I x f-ja od y

$$x=\frac{y^2}{4}$$

$$l=\int_0^4 \sqrt{1+x'^2(y)}\,dy$$

$$x'(y)=\frac{y}{2}$$

$$l=\int_0^4 \sqrt{1+\frac{y^2}{4}}\,dy=\left[\begin{array}{l|c|c} \frac{y}{2}=t & y & 0 & 4 \\ dy=2dt & t & 0 & 2 \end{array}\right]=$$

$$=2\int_0^2 \sqrt{1+t^2}\,dt$$ već rađeno parcijalnom int

$$l=2\sqrt{5}+\ln(2+\sqrt{5})$$